Typical Gibbs configurations for the 1d Random Field Ising Model with long range interaction.

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Typical Gibbs configurations for the 1d Random Field Ising Model with long range interaction. *

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Abstract We study a one–dimensional Ising spin systems with ferromagnetic, long–range interaction decaying as $n^{-2+\alpha}$, $\alpha \in [0, \frac{1}{2}]$, in the presence of external random fields. We assume that the random fields are given by a collection of symmetric, independent, identically distributed real random variables, gaussian or subgaussian with variance θ. We show that for temperature and variance of the randomness small enough, with an overwhelming probability with respect to the random fields, the typical configurations, within volumes centered at the origin whose size grow faster than any power of θ^{-1}, are intervals of + spins followed by intervals of − spins whose typical length is $\simeq \theta^{-\frac{2}{(1-2\alpha)}}$ for $0 \le \alpha < 1/2$ and $\simeq e^{\frac{1}{\theta^2}}$ for $\alpha = 1/2$.

1 Introduction

We consider a one dimensional ferromagnetic Ising model with a two body interaction $J(n) = n^{-2+\alpha}$ where n denotes the distance of the two spins and $\alpha \in [0, 1/2]$ tunes the decay of the interaction. We add to this term an external random field $h[\omega] := \{h_i[\omega], i \in \mathbb{Z}\}$ given by a collection of independent random variables, with mean zero, symmetrically distributed, variance θ, gaussian or sub–gaussian defined on a probability space $(\Omega, \Sigma, \mathbb{P})$. We study the magnetization profiles that are typical for the Gibbs measure when θ and the temperature are suitably small; this on a subspace $\Omega_1(\theta) \subset \Omega$ whose probability goes to 1 when $\theta \downarrow 0$.

A systematic and successful analysis of this model for $\theta = 0$ *i.e.* when the magnetic fields are absent has been already accomplished more than twenty years ago [21,10,11,12,13,14,15,1,16]. In particular it has been shown that it exhibits a phase transition only for $\alpha \in [0, 1)$. The presence of external random fields $(\theta \neq 0)$ modifies this picture. In [2], it has been proved that for $\alpha \in [0, 1/2]$ there exits an unique infinite volume Gibbs measure *i.e.* there is no phase transition. More recently in [8] it has been proved that when $\alpha \in (1/2, \frac{\log 3}{\log 2} - 1)$ the situation is analogous to the three dimensional short range random field Ising model [4] : for temperature and variance of the randomness small enough, there exist at least two distinct infinite volume Gibbs states, namely the μ^+ and the μ^- Gibbs states. The proof is based on the notion of contours introduced in [14] but using the geometrical description implemented in [5] better suited to describe the contribution of the random fields. A Peierls argument is obtained by using a lower bound of the deterministic part of the cost to erase a contour and controlling the stochastic part.

The method used in [2] to prove the uniqueness of the Gibbs measure is very powerful and general but does not provide any insight about the most relevant spin configurations of this measure.

In this paper we show that for temperature and variance of the randomness small enough the typical configurations are intervals of + spins followed by intervals of − spins whose typical length is $\theta^{-\frac{2}{(1-2\alpha)}}$ for $0 \le \alpha < 1/2$ and becomes exponentially large in terms of θ^{-2} for $\alpha = 1/2$. When $\theta > 0$ the Gibbs measures are random valued measures. We need therefore to localize the region in which we inspect the system. All our results are given uniformly for an increasing sequence of intervals, centered in one point, with a diameter

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going to infinity when $\theta \downarrow 0$.

The modifications induced by the presence of random fields has been already studied for one dimensional Kac model with range γ^{-1} [6,7,19]. In this case for θ and γ sufficiently small the typical size is γ^{-2}. The results are consistent if one recalls that the random field one dimensional Kac model exhibits a phase transition for $\gamma \downarrow 0$ and θ sufficiently small. In the present paper the typical size is obtained estimating suitable upper and lower bounds. The derivation of the upper bound is similar to the one used for the Kac model [6]. The lower bound follows from the observation that small intervals can be controlled with an estimate similar to those used in [8].

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2 Model, notations and main results

2.1. The model

Let $(\Omega, \mathcal{A}, I\!P)$ be a probability space on which we define $h \equiv \{h_i\}_{i \in \mathbb{Z}}$, a family of independent, identically distributed symmetric random variables. We assume that each h_i is Bernoulli distributed with $I\!P[h_i = +1] = I\!P[h_i = -1] = 1/2$. With minor modifications that will be mentioned we could also consider the cases of a Gaussian random variables with variance 1 or a subgaussian *i.e.* such that $I\!E[\exp(th_0)] \le \exp(t^2/2)\, \forall t \in I\!R$. This property is satisfied for example for $h_0 = X/a$ with X an uniform random variable on $[-a, +a]$, $a \in I\!R^+$ and up to an appropriate constant by any bounded symmetric random variable, see [17] for basic properties of sub–gaussian random variables.

The spin configurations space is $\mathcal{S} \equiv \{-1, +1\}^{\mathbb{Z}}$. If $\sigma \in \mathcal{S}$ and $i \in \mathbb{Z}$, σ_i represents the value of the spin at site i. The pair interaction among spins is given by $J(|i-j|)$ defined by

$$J(n) = \begin{cases} J(1) >> 1; \\ \dfrac{1}{n^{2-\alpha}} \quad \text{if} \quad n > 1, \quad \text{with } \alpha \in (-\infty, 1). \end{cases} \tag{2.1}$$

For $\Lambda \subseteq \mathbb{Z}$ we set $\mathcal{S}_\Lambda = \{-1, +1\}^\Lambda$; its elements are denoted by σ_Λ; also, if $\sigma \in \mathcal{S}$, σ_Λ denotes its restriction to Λ. Given $\Lambda \subset \mathbb{Z}$ finite, define

$$H_0(\sigma_\Lambda) = \frac{1}{2} \sum_{(i,j) \in \Lambda \times \Lambda} J(|i-j|)(1 - \sigma_i \sigma_j). \tag{2.2}$$

For $\omega \in \Omega$ set

$$G(\sigma_\Lambda)[\omega] := -\theta \sum_{i \in \Lambda} h_i[\omega] \sigma_i.$$

We consider the Hamiltonian given by the random variable on $(\Omega, \mathcal{A}, I\!P)$

$$H(\sigma_\Lambda)[\omega] = \frac{1}{2} \sum_{(i,j) \in \Lambda \times \Lambda} J(|i-j|)(1 - \sigma_i \sigma_j) + G(\sigma_\Lambda)[\omega]. \tag{2.3}$$

To take into account the interaction between the spins in Λ and those outside Λ we set for $\eta \in \mathcal{S}$

$$W(\sigma_\Lambda, \eta_{\Lambda^c}) = \sum_{i \in \Lambda} \sum_{j \in \Lambda^c} J(|i-j|)(1 - \sigma_i \eta_j) \tag{2.4}$$

and denote

$$H^{\eta}(\sigma_\Lambda)[\omega] := H(\sigma_\Lambda)[\omega] + W(\sigma_\Lambda, \eta_{\Lambda^c}). \tag{2.5}$$

In the following we drop out the ω from the notation. The corresponding *Gibbs measure* on the finite volume Λ, at inverse temperature $\beta > 0$ with boundary condition η is a random variable with values on the space of probability measures on $\mathcal{S}_\Lambda$ denoted by μ_Λ^η

$$\mu_\Lambda^\eta(\sigma_\Lambda) = \frac{1}{Z_\Lambda^\eta} \exp\{-\beta H^\eta(\sigma_\Lambda)\} \qquad \sigma_\Lambda \in \mathcal{S}_\Lambda, \tag{2.6}$$

where Z_Λ^η is the normalization factor. When the configuration η is taken so that $\eta_i = \tau$, $\tau = \pm 1$ for all $i \in \mathbb{Z}$ we denote the corresponding Gibbs measure by μ_Λ^+ when $\tau = 1$ and μ_Λ^- when $\tau = -1$. By FKG inequality the infinite volume limit $\Lambda \uparrow \mathbb{Z}$ of μ_Λ^+ and μ_Λ^- exists, say μ^+, μ^-. By a result of Aizenman and Wehr, see [2], *, when $\alpha \in [0, \frac{1}{2}]$ for $\mathbb{P}$–almost all ω, $\mu^+ = \mu^-$ and therefore there is an unique infinite volume Gibbs measure that will be denoted by μ.

2.2. Main result

Any spin configuration $\sigma \in \{-1, +1\}^{\mathbb{Z}}$ can be described in term of runs of $+1$, i.e. sequences of consecutive sites $i_1, i_1+1, i_1+2 \ldots \in \mathbb{Z}$ where $\sigma_k = +1, \forall k \in \{i_1, \ldots\}$, followed by runs of -1. A run could have length 1. To enumerate the runs we do as follows. Start from the site $i = 0$. Let $\sigma_0 = \tau$, $\tau \in \{-1, +1\}$ call $\mathcal{L}_1^\tau = \mathcal{L}_1^\tau(\sigma)$ the run containing the origin, $\mathcal{L}_2^{-\tau}$ the run on the right of $\mathcal{L}_1^\tau$ and $\mathcal{L}_0^{-\tau}$ the run on the left of $\mathcal{L}_1^\tau$. In this way to each configuration σ, we assign in a one to one way a sign $\tau = \sigma_0$ and a family of runs $(\mathcal{L}_j^{(-1)^{j+1}\tau}, i \in \mathbb{Z})$. To shorten notation we drop the $(-1)^{j+1}\tau$ and write simply $(\mathcal{L}_j, j \in \mathbb{Z})$.

Given a volume $V \subset \mathbb{Z}$ and a configuration σ_V, let $e_V = e_V(\sigma_V) = \sup(j \in \mathbb{Z} : \mathcal{L}_j \subset V)$ be the index of the rightmost run contained in V and $b_V = b_V(\sigma_V) = \inf(j \in \mathbb{Z} : \mathcal{L}_j \subset V)$ the index of the leftmost run contained in V. We consider the sequences of runs $(\mathcal{L}_j, b_V \le j \le e_V)$.

We give, in a volume V that we choose centered at the origin, in the regime β large and θ small, upper bound and lower bounds on the length of the runs.

In Theorem 2.1 we show that for volumes larger than any inverse power of θ up to subdominant terms with $\mathbb{P}$–probability larger than $1 - e^{-g(\theta)}$, where $g(\theta)$ is a function slowly going to infinity as $\theta \downarrow 0$, the typical configurations have runs with length of order $\theta^{-\frac{2}{1-2\alpha}}$ when $0 \le \alpha < 1/2$. When $\alpha = \frac{1}{2}$ we show in Theorem 2.2 that with overwhelming $\mathbb{P}$–probability the typical run that contains the origin is exponentially long in θ^{-2}.

Theorem 2.1 *Let $\alpha \in [0, \frac{1}{2})$ and $\zeta = \zeta(\alpha)$ as defined in (7.5), there exist $\theta_0 = \theta_0(\alpha)$, $\beta_0 = \beta_0(\alpha)$ and constants $c_i(\alpha)$, such that for all $0 < \theta \le \theta_0$, for all $\beta > \beta_0$*

$$\beta \ge \frac{\zeta}{2^8 \theta^2}, \tag{2.7}$$

if $0 < \alpha < 1/2$, setting $g(\theta) = (\log \frac{1}{\theta})(\log\log \frac{1}{\theta})$, with $\mathbb{P}$-probability larger than $1 - e^{-g(\theta)}$ and with a Gibbs measure larger than $1 - e^{-g(\theta)}$ the spin configurations are made of runs $(\mathcal{L}_j, b_V \le j \le e_V)$ satisfying

$$c_1(\alpha) \left(\log \frac{1}{\theta}\right)^{-\frac{2}{1-2\alpha}} \left(\log\log \frac{1}{\theta}\right)^{-\frac{1}{1-2\alpha}} \le \theta^{\frac{2}{1-2\alpha}} |\mathcal{L}_j| \le c_2(\alpha)(\log \frac{1}{\theta})(\log\log \frac{1}{\theta}), \tag{2.8}$$

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* A simplified proof of this result which avoids the introduction of metastates, by applying the FKG inequalities, is given by Bovier, see [3], chapter 7. Notice that although we assume that the distribution of the random field has isolated point masses, the result [2] still holds.

for all $j \in \{b_V, \dots e_V\}$ *where* V *is a volume centered at the origin having diameter*

$$\mathrm{diam}(V) = c_0(\alpha) e^{g(\theta)} \left(\frac{1}{\theta}\right)^{\frac{2}{1-2\alpha}}. \tag{2.9}$$

If $\alpha = 0$, $g(\theta)$ *has to be replaced by* $\hat{g}(\theta) = \log\left(\frac{\log\frac{1}{\theta}}{\theta}\right)$ *and (2.8) becomes*

$$c_1(0) \le \theta^2 |\mathcal{L}_i| \le c_2(0) \left(\log\frac{1}{\theta}\right)^3 \tag{2.10}$$

for all $j \in \{b_{\hat{V}}, \dots, e_{\hat{V}}\}$ *where* $\hat{V}$ *satisfies*

$$\mathrm{diam}(\hat{V}) = c_0(0) e^{\hat{g}(\theta)} \left(\frac{1}{\theta}\right)^2. \tag{2.11}$$

The proof of Theorem 2.1 follows from Propositions 3.1 and 4.1 and easy estimates.

Theorem 2.2 *For* $\alpha = 1/2$, *there exists* θ_0 *and* β_0 *and constants* c_i, *such that for all* $0 < \theta \le \theta_0$, *for all* $\beta > \beta_0$ *such that (2.7) is satisfied, the run that contains the origin, satisfies the inequalities*

$$\exp\frac{c_1}{\theta^2} \le |\mathcal{L}_1| \le \exp\frac{c_2}{\theta^2} \tag{2.12}$$

with $I\!P$*-probability larger than* $1 - e^{-\frac{c_0}{\theta^2}}$ *and with a Gibbs measure larger than* $1 - e^{-\frac{c_0}{\theta^2}}$.

Remark 2.3 . *The results for* $\alpha = 1/2$ *are less general because the probability estimates for the lower bound for* $\mathcal{L}_i$ *are not enough to extend results on exponential scales. However the estimates for the upper bound are true on a much larger scale, and we have results for a lot more than one run, see (3.5) and (3.6).*

3 The upper bound

Let $I \subset Z\!\!\!Z$ be an interval, $\tau = \pm 1$, denote

$$R^\tau(I) = \{\sigma \in \mathcal{S} : \sigma_i = \tau, \forall i \in I\} \tag{3.1}$$

the set of spin configurations equal to τ in the interval I and

$$R(I) := R^+(I) \cup R^-(I). \tag{3.2}$$

Let $L_{\max}$ be a positive integer and $V \subset Z\!\!\!Z$ be an interval centered at the origin with $|V| > L_{\max}$. Denote

$$\mathcal{R}(V, L_{\max}) = \bigcup_{I \subset V,\ |I| \ge L_{\max}} R(I), \tag{3.3}$$

the set of spin configurations having at least one run of $+1$ or -1 larger than $L_{\max}$ in V. The main result of this section is the following

Proposition 3.1 *Let $\alpha \in [0, \frac{1}{2}]$, there exist positive constants c_α and c'_α and $\theta_0 = \theta_0(\alpha)$ such that for all $\beta > 0$, for all decreasing real valued function $g_1(\theta) \geq 1$ defined on $I\!R$ that satisfies $\lim_{\theta \downarrow 0} g_1(\theta) = \infty$ there exist an $\Omega_3(\alpha) \subset \Omega$ with*

$$I\!P[\Omega_3(\alpha)] \geq \begin{cases} 1 - 2e^{-g_1(\theta)}, & \text{if } 0 \leq \alpha < \frac{1}{2}; \\ 1 - e^{-\frac{1}{2}e^{g_1(\theta)}}, & \text{if } \alpha = \frac{1}{2}, \end{cases} \tag{3.4}$$

$$L_{max}(\alpha) = \begin{cases} c'_\alpha g_1(\theta) \ \left(\frac{1}{\theta^2}\right)^{\frac{1}{1-2\alpha}}, & \text{if } 0 < \alpha < 1/2; \\ c'_0 g_1(\theta) \ \frac{1}{\theta^2}\left(\log\frac{1}{\theta}\right)^2, & \text{if } \alpha = 0; \\ c'_{1/2} e^{g_1(\theta)} \ e^{\frac{3}{2}\frac{8^2}{\theta^2}}(1+\frac{8}{\theta})^3, & \text{if } \alpha = 1/2, \end{cases} \tag{3.5}$$

and an interval $V(\alpha) \subset Z\!\!\!Z$ centered at the origin

$$|V(\alpha)| = \begin{cases} c'_\alpha e^{g_1(\theta)} \ \left(\frac{1}{\theta^2}\right)^{\frac{1}{1-2\alpha}}, & \text{if } 0 < \alpha < 1/2; \\ c'_0 e^{g_1(\theta)} \ \frac{1}{\theta^2}\left(\log\frac{1}{\theta}\right)^2, & \text{if } \alpha = 0; \\ c'_{1/2} e^{\frac{1}{2}\exp(g_1(\theta))} \ e^{\frac{8^2}{\theta^2}}\left(1+\frac{8}{\theta}\right)^3, & \text{if } \alpha = 1/2, \end{cases} \tag{3.6}$$

so that on $\Omega_3(\alpha)$, uniformly with respect to $\Lambda \subset Z\!\!\!Z$,

$$\sup_\eta \mu^\eta_\Lambda\left[\mathcal{R}(V(\alpha), L_{\max}(\alpha))\right] \leq \begin{cases} 2e^{g_1(\theta)} e^{-\beta c_\alpha \theta^{-\frac{2\alpha}{1-2\alpha}}}, & \text{if } 0 < \alpha < 1/2; \\ 2e^{g_1(\theta)} e^{-\beta c_0 \log\left(\frac{1}{\theta}\log\frac{1}{\theta}\right)}, & \text{if } \alpha = 0; \\ e^{\frac{1}{2}\exp(g_1(\theta))} \ e^{-\beta c_{1/2} e^{\frac{8^2}{2\theta^2}}}, & \text{if } \alpha = 1/2. \end{cases} \tag{3.7}$$

Remark:

There are various way to choose $g_1(\theta)$. If one is interested to get a good probability estimates in (3.4) and to have a volume $L_{max}(\alpha)$ not too much different from the $\theta^{-\frac{2}{1-2\alpha}}$ in the case $0 < \alpha < 1/2$, one can take for $g_1(\theta)$ a slowly varying function at zero. Note that $g_1(\theta) = (\log[1/\theta])(\log\log[1/\theta])$ have some advantages : $e^{-g_1(\theta)}$ decays faster than any inverse powers of θ^{-1}, the volume V grows faster than any polynomials in θ^{-1} and the asymptotic behavior of (3.7) is unaffected.

Proof: Since $I' \subset I$, $R(I) \subset R(I')$ we have

$$\bigcup_{I \subset V,\ |I| \geq L} R(I) \subset \bigcup_{I \subset V,\ |I| = L} R(I). \tag{3.8}$$

Therefore it is enough to consider the right hand side of (3.8) instead of the left hand one.

Assume that $I = \cup_{\ell=1}^M \Delta(\ell)$ where $\Delta(\ell)$, $\ell \in \{1, \ldots, M\}$, are adjacent intervals of length $|\Delta|$. We denote by Δ a generic interval $\Delta(\ell)$, $\ell \in \{1, \ldots, M\}$. We start estimating $\mu^\eta_\Lambda(R^+(\Delta))$. We bound from below Z^η_Λ by the sum over configurations constrained to be in $R^-(\Delta)$ and collect the contributions of the magnetic fields in Δ both in the numerator and in the denominator. We obtain:

$$\begin{aligned} \mu^\eta_\Lambda(R^+(\Delta)) &\leq \frac{\sum_{\sigma_\Lambda} e^{-\beta H^\eta(\sigma_\Lambda)[\omega]} 1\!\!1_{R^+(\Delta)}}{\sum_{\sigma_\Lambda} e^{-\beta H^\eta(\sigma_\Lambda)[\omega]} 1\!\!1_{R^-(\Delta)}} \\ &\leq e^{2\beta\theta \sum_{i\in\Delta} h_i[\omega]} \sup_{\sigma_{\Lambda\setminus\Delta}} \sup_{\eta_{\Lambda^c}} \frac{e^{-\beta[W(\sigma_\Delta,\sigma_{\Lambda\setminus\Delta})+W(\sigma_\Delta,\eta^c_\Lambda)]} 1\!\!1_{R^+(\Delta)}(\sigma_\Delta)}{e^{-\beta[W(\sigma_\Delta,\sigma_{\Lambda\setminus\Delta})+W(\sigma_\Delta,\eta^c_\Lambda)]} 1\!\!1_{R^-(\Delta)}(\sigma_\Delta} \\ &\leq e^{2\beta\theta \sum_{i\in\Delta} h_i[\omega]} e^{2\beta[\sum_{i\in\Delta}\sum_{j\in\Delta^c} J(|i-j|)]} \leq e^{2\beta\theta \sum_{i\in\Delta} h_i} e^{2\beta E_\alpha(|\Delta|)}. \end{aligned} \tag{3.9}$$

where $E_\alpha(|\Delta|)$ is defined by

$$E_\alpha(|\Delta|) = \begin{cases} 2(J(1)-1) + \frac{2|\Delta|^\alpha}{\alpha(1-\alpha)}, & \text{if } 0<\alpha<1; \\ 2(J(1)-1) + 2\log(|\Delta|)+4, & \text{if } \alpha = 0. \end{cases} \tag{3.10}$$

Calling

$$\Omega_1^-(\Delta) = \big\{\omega : \theta \sum_{i\in\Delta} h_i < -2E_\alpha(|\Delta|)\big\}, \tag{3.11}$$

on $\Omega_1^-(\Delta)$ we have

$$\sup_{\Lambda\subset\subset \mathbb{Z}} \sup_\eta \mu_\Lambda^\eta(R^+(\Delta)) \le e^{-2\beta E_\alpha(|\Delta|)}. \tag{3.12}$$

Define

$$\Omega_2^-(I) = \big\{\omega : \exists \ell_I^* \in \{1,\dots,M\} : \theta \sum_{i\in\Delta(\ell_I^*)} h_i < -2E_\alpha(|\Delta|)\big\}. \tag{3.13}$$

On $\Omega_2^-(I)$ we have

$$R^+(I) \subset R^+(\Delta(\ell_I^*)), \tag{3.14}$$

therefore, by (3.12),

$$\sup_{\Lambda\subset\subset \mathbb{Z}} \sup_\eta \mu_\Lambda^\eta(R^+(I)) \le e^{-2\beta E_\alpha(|\Delta|)}. \tag{3.15}$$

Assume $V = [-N|\Delta|, N|\Delta|]$. We can, then, cover V with overlapping intervals $I_k = [k|\Delta|, M|\Delta| + k|\Delta|[$ for $k \in \{-N,\dots,(N-M)\}$. It is easy to check that for any interval I of length $M|\Delta|$, $I\subset V$, there exists an unique $k\in\{-N,\dots,(N-M-1)\}$ such that

$$I \supset I_k \cap I_{k+1}. \tag{3.16}$$

Therefore one gets

$$\bigcup_{I\subset V,\, |I|=M|\Delta|} R^+(I) \subset \bigcup_{k=-N}^{N-M-1} \bigcup_{\substack{I: I_k\cap I_{k+1}\subset I\subset V \\ |I|=M|\Delta|}} R^+(I) \subset \bigcup_{k=-N}^{N-M-1} R^+(I_k\cap I_{k+1}). \tag{3.17}$$

Note that for all k there are $M-1$ consecutive blocks of size $|\Delta|$ in $I_k\cap I_{k+1}$ that will be indexed by $\ell_k \in \{2,\dots,M\}$. Define

$$\Omega_3^-(V) = \big\{\omega : \forall k \in \{-N,\dots,N-M\}, \exists \ell_k^* \in \{2,\dots,M\} :\ \theta \sum_{i\in\Delta(\ell_k^*)} h_i < -2E_\alpha(|\Delta|)\big\}. \tag{3.18}$$

If we notice that $R^+(I_k\cap I_{k+1}) \subset R^+(\Delta(\ell_k^*))$, it follows from (3.3), (3.17), and (3.15), that on $\Omega_3^-(V)$, uniformly with respect to $\Lambda\subset\mathbb{Z}$ we have

$$\sup_\eta \mu_\Lambda^\eta(R^+(V, M|\Delta|)) \le (2N+1)e^{-2\beta E_\alpha(|\Delta|)}. \tag{3.19}$$

Next we make a suitable choice of the parameters $|\Delta|, M, N$. Consider first the case $0<\alpha<1/2$.
Since the h_i are independent symmetric random variables, we have, see (3.11),

$$\mathbb{P}[\Omega_1^-(\Delta)] = \frac{1}{2}\Big(1 - \mathbb{P}\big[\big|\sum_{i\in\Delta} h_i\big| \le \frac{2E_\alpha(|\Delta|)}{\theta}\big]\Big) \equiv \frac{1}{2}(1-p_1), \tag{3.20}$$

$$\mathbb{P}[\Omega_2^-(I)] \geq 1 - \left(1 - \mathbb{P}[\Omega_1^-]\right)^M = 1 - \left(\frac{1+p_1}{2}\right)^M, \tag{3.21}$$

see (3.13), and, see (3.18),

$$\mathbb{P}[\Omega_3^-(V)] \geq 1 - (2N+1)\left(\frac{1+p_1}{2}\right)^{M-1}. \tag{3.22}$$

To estimate p_1, we apply the following estimate, see Le Cam [18], pg 407, which holds for i.i.d. random variables, symmetric and subgaussian:

$$\sup_{x \in \mathbb{R}} \mathbb{P}[\sum_{i=1}^{|\Delta|} h_i \in [x, x+\tau]] \leq \frac{2\sqrt{\pi}}{\sqrt{|\Delta| \mathbb{E}[1 \wedge (h_1/\tau)^2]}}. \tag{3.23}$$

When $\{h_i, i \in \mathbb{Z}\}$ have symmetric Bernoulli distribution, assuming that $\tau \geq 1$, one has $\mathbb{E}[(h_1/\tau)^2 \mathbb{1}_{|h_1| \leq \tau}] \geq \tau^{-2}$. For random fields having different distribution see Remark 3.2.

For any $0 < B < 1$, take Δ such that $p_1 \leq B < 1$ and $\tau = 2E_a(|\Delta|)/\theta \geq 1$. Assuming that the second constraint holds and using (3.23), to satisfy the first constraint, it is enough that

$$p_1 \leq \frac{8E_\alpha(|\Delta|)\sqrt{\pi}}{\theta\sqrt{|\Delta|}} \leq B. \tag{3.24}$$

We choose

$$|\Delta| = \left(\frac{32}{B\theta\alpha(1-\alpha)}\right)^{\frac{2}{1-2\alpha}}. \tag{3.25}$$

Then it is easy to check that there exists a $\theta_0 = \theta_0(\alpha, J(1))$ but independent on B such that (3.24) and $\tau \geq 1$ are satisfied for all $0 < \theta \leq \theta_0$. Choosing

$$M = \frac{2g_1(\theta)}{\log \frac{2}{1+B}} \tag{3.26}$$

and

$$2N+1 = e^{g_1(\theta)}\frac{1+B}{2} \tag{3.27}$$

with $g_1(\theta)$ so that $\lim_{\theta \downarrow 0} g_1(\theta) = \infty$, (3.4), (3.5), (3.6), and (3.7) are proven for $0 < \alpha < 1/2$. The actual value of B affects only the values of the constants.
When $\alpha = 0$, Le Cam inequality suggests

$$|\Delta| = \theta^{-2}\left(\frac{64\sqrt{\pi}}{B}\log\theta^{-1}\right)^2. \tag{3.28}$$

Taking M and N as in (3.26) and (3.27), one gets (3.4), (3.5), (3.6), and (3.7).
When $\alpha = 1/2$

$$\Omega_1(\Delta) = \{\omega : \theta\sum_{i \in \Delta} h_i \leq -8\sqrt{\Delta}\}. \tag{3.29}$$

Le Cam inequality is useless. We use the Berry-Esseen Theorem [9] that gives

$$\mathbb{P}[\Omega_1(\Delta)] \geq \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{-\frac{8}{\theta}} e^{-\frac{x^2}{2}}\,dx - \frac{C_{BE}}{\sqrt{\Delta}} \tag{3.30}$$

where $C_{BE} \leq 7.5$ is the Berry-Esseen constant. By the lower bound $\int_{-\infty}^{-y} e^{-\frac{x^2}{2}}\, dx \geq \frac{y}{1+y^2} e^{-\frac{1}{2}y^2}$, we have

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-\frac{8}{\theta}} e^{-\frac{x^2}{2}}\, dx \geq \frac{1}{\sqrt{2\pi}} \frac{1}{1+\frac{8}{\theta}} e^{-\frac{8^2}{2\theta^2}}. \tag{3.31}$$

Choosing

$$\Delta = 16^2(2\pi)\left(1+\frac{8}{\theta}\right)^2 e^{\frac{8^2}{\theta^2}}, \tag{3.32}$$

so that the right hand side of (3.30) is strictly positive,

$$M = 2\sqrt{2\pi}(1+\frac{8}{\theta})e^{\frac{8^2}{2\theta^2}} e^{g_1(\theta)}, \tag{3.33}$$

and

$$2N+1 = e^{\frac{1}{2}e^{g_1(\theta)}} \tag{3.34}$$

we get (3.4), (3.5), (3.6), and (3.7).

■

Remark 3.2 . *To apply (3.23), one needs a lower bound for the censored variance at τ of h_1 which is $I\!E[1 \wedge (h_1/\tau)^2]$. A simple one is $I\!E[(h_1/\tau)^2 1\!\!1_{|h_1| \leq \tau}]$ which is bounded from below by half the variance of h_1 times τ^{-2} by taking τ large enough. However one can also get more precise bound since the difference between the censored variance and the variance can be estimated by using an exponential Markov inequality that can be obtained as a consequence of the definition of sub-gaussian. When $h_i, i \in Z\!\!\!Z$ are normal distributed the bound (3.23) can be easily improved to*

$$\sup_{x \in I\!R} I\!P[\sum_{i=1}^{|\Delta|} h_i \in [x, x+\tau]] \leq \frac{\tau}{\sqrt{2\pi|\Delta|}}. \tag{3.35}$$

4 Lower bound

Let $\Delta \subset Z\!\!\!Z$ be an interval, $\partial\Delta = \{i \in Z\!\!\!Z : d(i,\Delta) = 1\}$, $\tau = \pm 1$, define

$$\mathcal{W}(\Delta, \tau) = \{\sigma \in \mathcal{S} : \sigma_i = \tau, \forall i \in \Delta, \sigma_{\partial\Delta} = -\tau\}. \tag{4.1}$$

Let $L_{\min}$ be a positive integer and $V \subset Z\!\!\!Z$ be an interval centered at the origin, with $|V| > L_{\min}$. We denote for $i \in V$ and $\tau \in \{-1, +1\}$,

$$\nu_i(L_{\min}, \tau) = \bigcup_{\Delta \ni i,\, |\Delta| \leq L_{\min}} \mathcal{W}(\Delta, \tau), \tag{4.2}$$

$$\mathcal{V}(V, L_{\min}) = \bigcup_{i \in V} \left[\nu_i(L_{\min}, +) \cup \nu_i(L_{\min}, -)\right]. \tag{4.3}$$

The main result of this section is the following.

Proposition 4.1 *Let $\alpha \in [0, \frac{1}{2}]$, $\theta > 0$, $\zeta = \zeta(\alpha)$ as defined in (7.5). There exists $\theta_0 = \theta_0(\alpha)$ and $\beta_0 = \beta_0(\alpha)$ such that for $0 < \theta < \theta_0$ and $\beta > \beta_0$, for all $D > 1$, for all decreasing real valued function $g_2(x)$ defined on $I\!R^+$ such that $\lim_{x\downarrow 0} g_2(0) = \infty$ but $\lim_{x\downarrow 0} \frac{g_2(x)}{x} = 0$, if we denote*

$$\bar{b} := \min(\frac{\beta\zeta}{4}, \frac{\zeta^2}{2^{10}\theta^2}) \tag{4.4}$$

then there exists $\Omega_5(\alpha) \subset \Omega$ with

$$I\!P[\Omega_5(\alpha)] \geq \begin{cases} 1 - 5\left(\bar{b}\right)^{\frac{2}{(1-2\alpha)}} e^{-(4D-1)g_2(\bar{b})}, & \text{if } 0 < \alpha < 1/2; \\ 1 - 5\left(\frac{\bar{b}}{g_2(\bar{b})}\right)^2 \left(4 + \log\left[\frac{\bar{b}}{8g_2(\bar{b})}\right]\right)^2 e^{-(4D-1)g_2(\bar{b})}, & \text{if } \alpha = 0; \\ 1 - e^{-g_2(\bar{b})}, & \text{if } \alpha = 1/2. \end{cases} \tag{4.5}$$

For

$$L_{min}(\alpha) = \begin{cases} \left(\frac{\bar{b}}{Dg_2(\bar{b})}\right)^{\frac{1}{1-2\alpha}} \left(\frac{1}{4+\log(\bar{b})^{\frac{1}{1-2\alpha}}}\right)^{\frac{1}{1-2\alpha}}, & \text{if } 0 < \alpha < 1/2; \\ \frac{\bar{b}}{Dg_2(\bar{b})}\left(4 + \log\left[\frac{\bar{b}}{Dg_2(\bar{b})}\right]\right), & \text{if } \alpha = 0; \\ e^{\frac{\bar{b}}{2D}-4}, & \text{if } \alpha = 1/2, \end{cases} \tag{4.6}$$

and

$$V_{min}(\alpha) = \begin{cases} e^{g_2(\bar{b})}(\bar{b})^{\frac{1}{1-2\alpha}}, & \text{if } 0 < \alpha < 1/2; \\ e^{g_2(\bar{b})}\frac{\bar{b}}{Dg_2(\bar{b})}\left(4 + \log\left[\frac{\bar{b}}{Dg_2(\bar{b})}\right]\right), & \text{if } \alpha = 0; \\ e^{\frac{\bar{b}}{2}(1-\frac{1}{D})}e^{-2g_2(\bar{b})}, & \text{if } \alpha = 1/2, \end{cases} \tag{4.7}$$

on $\Omega_5(\alpha)$, for all $\Lambda \subset Z\!\!Z$ large enough,

$$\mu^+_\Lambda[\mathcal{V}(V_{min}(\alpha), L_{min}(\alpha))] \leq \begin{cases} 5\left(\bar{b}\right)^{\frac{2}{(1-2\alpha)}} e^{-(4D-1)g_2(\bar{b})}, & \text{if } 0 < \alpha < 1/2; \\ 5\left(\frac{\bar{b}}{8Dg_2(\bar{b})}\right)^2 \left(4 + \log\left[\frac{\bar{b}}{Dg_2(\bar{b})}\right]\right)^2 e^{-(4D-1)g_2(\bar{b})}, & \text{if } \alpha = 0; \\ e^{-g_2(\bar{b})}, & \text{if } \alpha = 1/2. \end{cases} \tag{4.8}$$

Remark: The estimate (4.8) is uniform in Λ, therefore by the uniqueness of the infinite volume Gibbs measure, [2], Proposition 4.1 holds for the infinite volume Gibbs measure μ.

Proof: Since the boundary conditions are homogeneous equal to + we apply the geometrical description of the spin configuration presented in [5]. In the following we will assume that the notions of triangles, contours, and their properties are known to the reader. In Section 7 we summarize definitions and main properties used in the proof. Let $\mathcal{T} = \{\underline{T}\}$ be the set of families of triangles compatible with the chosen + boundary conditions on Λ. Let denote by $|T|$ the *mass* of the triangle T, *i.e.* the cardinality of $T \cap Z\!\!Z$, see (7.1). It is convenient to identify in $\underline{T} \in \mathcal{T}$ families of triangles having the same mass,

$$\underline{T} = \{\underline{T}^{(1)}, \ldots, \underline{T}^{(k_{\underline{T}})}\}, \tag{4.9}$$

arranged in increasing order, where $k_{\underline{T}} = \sup\{|T| : T \in \underline{T}\} \in I\!N$ and for $\ell \in \{1, \ldots, k_{\underline{T}}\}$, $\underline{T}^{(\ell)}$ is the family of $n_\ell \equiv n_\ell(\underline{T}) \in I\!N$ triangles in $\underline{T}$ having all the mass ℓ. By convention $n_\ell(\underline{T}) = 0$ when there is no triangle of mass ℓ in $\underline{T}$. We denote

$$|\underline{T}|^x = \sum_{\ell=1}^{k_{\underline{T}}} n_\ell(\underline{T})\,\ell^x,\ x \in I\!R,\ x \neq 0, \tag{4.10}$$

and

$$\log |\underline{T}| = \sum_{\ell=1}^{k_{\underline{T}}} n_\ell(\underline{T})(4+\log \ell). \tag{4.11}$$

Let $\Lambda \subset \mathbb{Z}$ be an interval large enough, $V \subset \Lambda$ and L an integer, $L \leq |V|$. Since $\mu_\Lambda^+(\cup_{i\in V}\nu_i(L,-)) \leq \sum_{i\in V}\mu_\Lambda^+(\nu_i(L,-))$, it is enough to estimate for a given $i \in V$, $\mu_\Lambda^+(\nu_i(L,-))$. Applying (4.2) one has

$$\mu_\Lambda^+(\nu_i(L,-)) \leq \sum_{\ell_0=1}^{L} \sum_{\Delta:\Delta\ni i, |\Delta|=\ell_0} \mu_\Lambda^+(\mathcal{W}(\Delta,-)). \tag{4.12}$$

It remains to estimate $\mu_\Lambda^+(\mathcal{W}(\Delta,-))$, for a given $i \in V$, $\Delta \ni i$ and $|\Delta| = \ell_0$. We denote by

$$\mathcal{C} = \mathcal{C}(\Delta,-) = \{\underline{T} \in \mathcal{T} \text{compatible with } \mathcal{W}(\Delta,-)\}. \tag{4.13}$$

A family $\underline{T}$ is said compatible with the event $\mathcal{W}(\Delta,-)$ if $\underline{T}$ corresponds to a spin configuration where the event $\mathcal{W}(\Delta,-)$ occurs. By construction the families of triangles in $\mathcal{C}$ satisfy only one of the two following conditions:

- there exists $T_0 \in \mathcal{C}$ so that $\Delta = supp(T_0)$
- there exist two triangles $T_{right} = T_{right}(\Delta)$ and $T_{left} = T_{left}(\Delta)$ one on the right and one on the left of Δ that are adjacent * to Δ.

The fact that T_{left} (resp. T_{right}) is on the left (resp. right) of Δ and is adjacent to it will be denoted by $T_{left} \triangleleft \Delta$, (resp $T_{right} \triangleright \Delta$). By (7.2) $\ell_0 = \mathrm{dist}(T_{left}, T_{right}) \geq |T_{right}| \wedge |T_{left}|$, *i.e.* at least one of the two triangles (T_{left}, T_{right}) has support smaller or equal than ℓ_0. We make the partition:

$$\mathcal{C} = \cup_{j=1}^3 \mathcal{A}_j \tag{4.14}$$

where $\mathcal{A}_j = \mathcal{A}_j(\Delta, i)$ are defined by:

$$\mathcal{A}_1 = \{\underline{T} \in \mathcal{C} : \exists T_0 \in \underline{T}, supp(T_0) = \Delta\}; \tag{4.15}$$

$$\mathcal{A}_2 = \cup_{\ell=1}^{\ell_0}\mathcal{A}_2(\ell) \text{ with } \mathcal{A}_2(\ell) = \{\underline{T} \in \mathcal{C} : \exists T_{left} \in \underline{T},\, T_{left} \triangleleft \Delta,\, |T_{left}| = \ell\}; \tag{4.16}$$

$$\mathcal{A}_3 = \cup_{\ell=1}^{\ell_0}\mathcal{A}_3(\ell) \text{ with } \mathcal{A}_3(\ell) = \{\underline{T} \in \mathcal{C}\setminus\mathcal{A}_2 : \exists T_{right} \in \underline{T},\, T_{right} \triangleright \Delta,\, |T_{right}| = \ell\}. \tag{4.17}$$

Any family in $\mathcal{A}_1$ can be written as $(T_0, \underline{T}) \in \mathcal{A}_1$ where $T_0 \notin \underline{T}$. We denote by $\mathcal{A}_1 \setminus T_0$ the set all these $\underline{T}$ such that $(T_0, \underline{T}) \in \mathcal{A}_1$, with the same meaning we denote $\mathcal{A}_2(\ell)\setminus T_{left}$ and $\mathcal{A}_3(\ell)\setminus T_{right}$. We have

$$\begin{aligned}\mu_\Lambda^+(\mathcal{W}(\Delta,-)) = &\sum_{\underline{T}\in\mathcal{A}_1\setminus T_0} \mu_\Lambda^+(T_0 \cup \underline{T}) 1\!\!1_{\{Supp(T_0)=\Delta\}} \\ &+ \sum_{\ell=1}^{\ell_0} \sum_{T_{left}:|T_{left}|=\ell} 1\!\!1_{\{T_{left}\triangleleft\Delta\}} \sum_{\underline{T}\in\mathcal{A}_2(\ell)\setminus T_{left}} \mu_\Lambda^+(T_{left}\cup\underline{T}) \\ &+ \sum_{\ell=1}^{\ell_0} \sum_{T_{right}:|T_{right}|=\ell} 1\!\!1_{\{T_{right}\triangleright\Delta\}} \sum_{\underline{T}\in\mathcal{A}_3(\ell)\setminus T_{right}} \mu_\Lambda^+(T_{right}\cup\underline{T}).\end{aligned} \tag{4.18}$$

* We say that T is adjacent to an interval Δ if $0 < d(supp(T), \Delta) < 1$. *i.e.* $\Delta \cap supp(T) = \emptyset$ and T is the first triangle on the right or the left of Δ having the support at distance from Δ smaller than 1.

For any given triangle T, with $|T| = \ell$, recalling the definition of contours in Section 7, let

$$\mathcal{A}(T) \equiv \mathcal{A}(T,\ell) = \{\underline{S} \in \mathcal{T} : T \notin \underline{S}\,; (T,\underline{S}) \text{ form a contour}\,;\ \forall S \in \underline{S},\ |S| < \ell\}. \tag{4.19}$$

Remark 4.2 . All the triangles belonging to $\mathcal{A}(T,\ell)$ have mass $\ell_1 < \ell$ and form a contour with T. Notice that triangles T_1 with $|T_1| = \ell_1$, $\ell_1 < \ell$ might belong to the same contour Γ of T but when we remove the triangles in Γ different than T, having support larger or equal to ℓ the resulting family might not form a single contour with T. These triangles are not in $\mathcal{A}(T,\ell)$.

We start analyzing the first term on the right hand side of (4.18). We decompose $\underline{T} \in \mathcal{A}_1 \setminus T_0$ as $\underline{S}_1 \cup \underline{T}'$ with $\underline{S}_1 \in \mathcal{A}(T_0,\ell_0)$ and $\underline{T}' \notin \mathcal{A}(T_0,\ell_0)$, obtaining

$$\begin{aligned}\sum_{\underline{T}\in\mathcal{A}_1\setminus T_0} \mu^+_\Lambda(T_0 \cup \underline{T}) &= \sum_{\underline{S}_1\sim T_0} \mathbb{I}_{\{\underline{S}_1\in\mathcal{A}(T_0,\ell_0)\}} \sum_{\underline{T}'\sim(T_0\cup\underline{S}_1)} \mathbb{I}_{\{\underline{T}'\notin\mathcal{A}(T_0,\ell_0)\}}\mu^+_\Lambda(T_0\cup\underline{S}_1\cup\underline{T}') \\ &= \sum_{\underline{S}_1\sim T_0} \mathbb{I}_{\{\underline{S}_1\in\mathcal{A}(T_0,\ell_0)\}}\mu^+_\Lambda(T_0\cup\underline{S}_1) = \sum_{\underline{S}_1\in\mathcal{A}(T_0,\ell_0)} \mu^+_\Lambda(T_0\cup\underline{S}_1).\end{aligned} \tag{4.20}$$

Recall that $\underline{S}_1 \sim T_0$ means that $\underline{S} \cup T_0$ is an allowed configuration of triangles. Applying the same decomposition for the remaining two terms on the right hand side of (4.18) we get

$$\begin{aligned}\mu^+_\Lambda(\mathcal{W}(\Delta,-)) = &\sum_{\underline{S}_1\in\mathcal{A}(T_0,\ell_0)} \mu^+_\Lambda(T_0\cup\underline{S}_1)\mathbb{I}_{\{supp(T_0)=\Delta\}} \\ &+\sum_{\ell=1}^{\ell_0}\sum_{T_{left}:|T_{left}|=\ell} \mathbb{I}_{\{T_{left}\,\triangleleft\,\Delta\}} \sum_{\underline{S}_1\in\mathcal{A}(T_{left},\ell)} \mu^+_\Lambda(T_{left}\cup\underline{S}_1) \\ &+\sum_{\ell=1}^{\ell_0}\sum_{T_{right}:|T_{right}|=\ell} \mathbb{I}_{\{T_{right}\,\triangleright\,\Delta\}} \sum_{\underline{S}_1\in\mathcal{A}(T_{right},\ell)} \mu^+_\Lambda(T_{right}\cup\underline{S}_1).\end{aligned} \tag{4.21}$$

We estimate separately each term in the previous sums. They are all alike $\mu^+_\Lambda(T\cup\underline{S})$ with $\underline{S}\in\mathcal{A}(T,\ell)$ see (4.19) and $|T| = \ell$. Recalling (4.9), we identify in $\underline{S}$ the families of triangles having the same mass. By construction we have $k_{\underline{S}} \in \{1,\ldots,\ell-1\}$. We follow an argument used in [8] which consists of 4 steps. We consider first the case $0 < \alpha < 1/2$, the case $\alpha = 0$ and $\alpha = \frac{1}{2}$ will be discussed later.

Step I

For each $j = \{1,\ldots,k_{\underline{S}}\}$ we extract a term $\sum_{k=1}^{j} n_k(\underline{S})k^\alpha$ from the deterministic part of the Hamiltonian, *i.e.* using Theorem 7.3, we write

$$\begin{aligned}\mu^+_\Lambda(T\cup\underline{S}) =& \frac{1}{Z^+_\Lambda[\omega]}\sum_{\underline{T}'\sim T\cup\underline{S}} e^{-\beta H^+(\underline{T}'\cup T\cup\underline{S})[\omega]} \\ &\le e^{-\beta\frac{\zeta}{2}(\sum_{k=1}^{j} n_k(\underline{S})k^\alpha)}\frac{1}{Z^+_\Lambda[\omega]}\sum_{\underline{T}'\sim T\cup\underline{S}} e^{-\beta H^+_0(\underline{T}'\cup T\cup\underline{S}\setminus(\cup_{k=1}^{j}\underline{S}^{(k)})+\beta\theta G(\sigma(\underline{T}'\cup T\cup\underline{S}))[\omega]}.\end{aligned} \tag{4.22}$$

We add to this list of $k_{\underline{S}}$ inequalities a $k_{\underline{S}}+1$–th inequality that we get when, after extracting all the terms corresponding to $\underline{S}$, we extract the term corresponding to T *i.e.*

$$\mu^+_\Lambda(T\cup\underline{S}) \le e^{-\beta\frac{\zeta}{2}(\sum_{k=1}^{k_{\underline{S}}} n_k(\underline{S})k^\alpha+\ell^\alpha)}\frac{1}{Z^+_\Lambda[\omega]}\sum_{\underline{T}'\sim T\cup\underline{S}} e^{-\beta H^+_0(\underline{T}')+\beta\theta G(\sigma(\underline{T}'\cup\underline{S}\cup T))[\omega]}. \tag{4.23}$$

Observing the right hand side of (4.22) and (4.23), one notes that the H_0^+ and G are not evaluated at the same configuration of triangles. In the next step we compensate this discrepancy by a corrective term.

Step II

For each $j \in \{1,\dots,k_{\underline{S}}\}$ we multiply and divide (4.22) by

$$\sum_{\underline{T}'\sim T\cup \underline{S}} e^{-\beta H_0^+(\underline{T}'\cup T\cup \underline{S}\setminus(\cup_{\ell=1}^{j}\underline{S}^{(\ell)})+\beta\theta G(\sigma(\underline{T}'\cup T\cup \underline{S}\setminus(\cup_{\ell=1}^{j}\underline{S}^{(\ell)}))[\omega]} \tag{4.24}$$

and when $j = k_{\underline{S}}+1$, see (4.23) by

$$\sum_{\underline{T}'\sim T\cup \underline{S}} e^{-\beta H_0^+(\underline{T}')+\beta\theta G(\sigma(\underline{T}'))[\omega]}. \tag{4.25}$$

Setting for $j \in \{1,\dots,k_{\underline{S}}\}$

$$F_j[\omega] := \frac{1}{\beta}\ln\left\{\frac{\sum_{\underline{T}'\sim T\cap \underline{S}} e^{-\beta H_0^+(\underline{T}'\cup T\cup \underline{S}\setminus(\cup_{\ell=1}^{j}\underline{S}^{(\ell)})+\beta\theta G(\sigma(\underline{T}'\cup T\cup \underline{S}))[\omega]}}{\sum_{\underline{T}'\sim T\cup \underline{S}} e^{-\beta H_0^+(\underline{T}'\cup T\cup \underline{S}\setminus(\cup_{\ell=0}^{j}\underline{S}^{(\ell)})+\beta\theta G(\sigma(\underline{T}'\cup T\cup \underline{S}\setminus(\cup_{\ell=1}^{j}\underline{S}^{(\ell)}))[\omega]}}\right\}, \tag{4.26}$$

and for $j = k_{\underline{S}}+1$

$$F_{k_{\underline{S}}+1}[\omega] = \frac{1}{\beta}\ln\left\{\frac{\sum_{\underline{T}'\sim T\cup \underline{S}} e^{-\beta H_0^+(\underline{T}')+\beta\theta G(\sigma(\underline{T}\cup T\cup \underline{S}))[\omega]}}{\sum_{\underline{T}'\sim T\cup \underline{S}} e^{-\beta H_0^+(\underline{T}')+\beta\theta G(\sigma(\underline{T}'))[\omega]}}\right\} \tag{4.27}$$

we have the following set of inequalities: for $j \in \{1,\dots,k_{\underline{T}}+1\}$

$$\mu_\Lambda^+(T\cup \underline{S}) \le e^{-\beta\frac{\zeta}{2}(\sum_{\ell=1}^{j} n_\ell(\underline{S})\ell^\alpha)+\beta F_j[\omega]}\mu_\Lambda^+(T\cup \underline{S}\setminus(\cup_{\ell=1}^{j}\underline{S}^{(\ell)})) \le e^{-\beta\frac{\zeta}{2}(\sum_{k=1}^{j} n_k(\underline{S})k^\alpha)+\beta F_j[\omega]}. \tag{4.28}$$

Step III

We make a partition of the probability space to take into account the fluctuations of the F_i in (4.28). For each $(T,\underline{S})$ we write

$$\Omega = \cup_{j=0}^{k_{\underline{S}}+1} B_j, \tag{4.29}$$

where, recalling (4.10), for $j \in \{1,\dots,k_{\underline{S}}\}$

$$B_j = B_j((T,\underline{S})) = \{\omega : F_j[\omega] \le \frac{\zeta}{4}\sum_{k=1}^{j} n_k(\underline{S})\,k^\alpha,\ \text{and for } \forall i\in\{j+1,\dots \ell_0\}, F_i[\omega] > \frac{\zeta}{4}\sum_{k=1}^{i} n_k(\underline{S})\,k^\alpha\}; \tag{4.30}$$

$$B_{k_{\underline{S}}+1} = B_{k_{\underline{S}}+1}((T,\underline{S})) = \left\{\omega : F_{k_{\underline{S}}+1}[\omega] \le \frac{\zeta}{4}\left(\sum_{k=1}^{k_{\underline{S}}} n_k(\underline{S})\,k^\alpha + \ell^\alpha\right)\right\}; \tag{4.31}$$

$$B_0 = B_0((T,\underline{S})) = \{\omega : \forall i\in\{1,\dots,k_{\underline{S}}+1\}, F_i[\omega] > \frac{\zeta}{4}\sum_{k=1}^{i} n_k(\underline{S})\,k^\alpha\}. \tag{4.32}$$

The point is that using exponential inequalities for Lipschitz function of subgaussian random variables, see [8] Section 4 for details, one has : for all $\alpha\in(0,1)$ For $0\le j\le k_{\underline{S}}$,

$$I\!E\left[1\!\!1_{B_j}\right] \le e^{-\frac{\zeta^2}{2^{10}\theta^2}\left(\sum_{k=j+1}^{k_{\underline{S}}} n_k(\underline{S})\,k^{2\alpha-1}+\ell^{2\alpha-1}\right)}. \tag{4.33}$$

with the convention that an empty sum is zero. For $j = k_{\underline{S}} + 1$ we use $I\!\!E\left[1\!\!1_{B_{k_{\underline{S}}+1}}\right] \le 1$.

Step IV

Using (4.29), we have

$$I\!\!E\left[\mu_\Lambda^+(T \cup \underline{S})\right] = \sum_{j=0}^{k_{\underline{S}}+1} I\!\!E\left[\mu_\Lambda^+(T \cup \underline{S}) 1\!\!1_{\{B_j\}}\right], \tag{4.34}$$

then, (4.28) entails

$$I\!\!E\left[\mu_\Lambda^+(T \cup \underline{S}) 1\!\!1_{\{B_j\}}\right] \le e^{-\beta\frac{\zeta}{2}(\sum_{k=1}^{j} n_k(\underline{S})\, k^\alpha)} I\!\!E\left[e^{\beta F_j} 1\!\!1_{\{B_j\}}\right]. \tag{4.35}$$

Recalling (4.30),(4.31) and (4.32), on B_j we have

$$F_j \le \frac{\zeta}{4} \sum_{k=1}^{j} n_k(\underline{S})\, k^\alpha \tag{4.36}$$

that gives with (4.35) and (4.33)

$$I\!\!E\left[\mu_\Lambda^+(T \cup \underline{S}) 1\!\!1_{\{B_j\}}\right] \le e^{-\beta\frac{\zeta}{4}\sum_{k=1}^{j} n_k(\underline{S})\, k^\alpha} e^{-\frac{\zeta^2}{2^{10}\theta^2}\left(\sum_{k=j+1}^{k_{\underline{S}}} n_k(\underline{S})\, k^{2\alpha-1} + \ell^{2\alpha-1}\right)}. \tag{4.37}$$

Coming back to (4.34) we get

$$\begin{aligned} I\!\!E\left[\mu_\Lambda^+(T \cup \underline{S})\right] &\le \sum_{j=0}^{k_{\underline{S}}} e^{-\frac{\beta\zeta}{4}\sum_{k=1}^{j} n_k(\underline{S})\, k^\alpha} e^{-\frac{\zeta^2}{2^{10}\theta^2}\left(\sum_{k=j+1}^{k_{\underline{S}}} n_k(\underline{S})\, k^{2\alpha-1} + \ell^{2\alpha-1}\right)} + e^{-\frac{\beta\zeta}{4}\left(\sum_{k=1}^{k_{\underline{S}}} |\underline{S}^{(k)}| n_k(\underline{S})\, k^\alpha + \ell^\alpha\right)} \\ &\le (k_{\underline{S}} + 2) e^{-\bar{b}\left(\sum_{k=1}^{k_{\underline{S}}} n_k(\underline{S})\, k^{2\alpha-1} + \ell^{2\alpha-1}\right)}, \end{aligned} \tag{4.38}$$

where

$$\bar{b} = \min\left(\frac{\beta\zeta}{4}, \frac{\zeta^2}{2^{10}\theta^2}\right). \tag{4.39}$$

Final conclusions To estimate (4.12) we take into account the partition done in (4.21). Corresponding to the first term in (4.21), using (4.38) and (4.15), we have for each $i \in V$

$$\begin{aligned} I_1(i) &\equiv \sum_{\ell_0=1}^{L} \sum_{\Delta:\Delta \ni i, |\Delta|=\ell_0} \sum_{\underline{S}_1 \in \mathcal{A}(T_0,\ell_0)} I\!\!E\left[\mu_\Lambda^+(T_0 \cup \underline{S}_1)\right] 1\!\!1_{\{supp T_0 = \Delta\}} \\ &= \sum_{\ell_0=1}^{L} \sum_{T_0:T_0 \ni i, |T_0|=\ell_0} \sum_{\underline{S}_1 \in \mathcal{A}(T_0,\ell_0)} I\!\!E\left[\mu_\Lambda^+(T_0 \cup \underline{S}_1)\right] \\ &\le \sum_{\ell_0=1}^{L} \sum_{T_0:T_0 \ni i, |T_0|=\ell_0} \sum_{\underline{S}_1 \in \mathcal{A}(T_0,\ell_0)} (\ell_0 + 2) e^{-\bar{b}\left(\sum_{k=1}^{k_{\underline{S}_1}} n_k(\underline{S}_1)\, k^{2\alpha-1} + \ell_0^{2\alpha-1}\right)}. \end{aligned} \tag{4.40}$$

Since all the triangles in $\underline{S}_1 \in \mathcal{A}(T_0, \ell_0)$ are smaller than ℓ_0, we have

$$\sum_{k=1}^{k_{\underline{S}_1}} n_k(\underline{S}_1)\, k^{2\alpha-1} + \ell_0^{2\alpha-1} \ge \frac{1}{\ell_0^{1-2\alpha}(4 + \log \ell_0)} \left(\sum_{k=1}^{k_{\underline{S}_1}} n_k(\underline{S}_1)(4 + \log k) + (4 + \log \ell_0)\right) \tag{4.41}$$

so that from (4.40) we have

$$
\begin{aligned}
I_1(i) &\le \sum_{\ell_0=1}^{L} (\ell_0+2) \sum_{T_0:T_0\ni i,|T_0|=\ell_0} \;\sum_{\underline{S}_1\in\mathcal{A}(T_0,\ell_0)} e^{-\bar b\left(\frac{1}{\ell_0^{1-2\alpha}(4+\log \ell_0)}\left(\sum_{k=1}^{k_{\underline{S}_1}} n_k(\underline{S}_1)(4+\log k)+(4+\log \ell_0)\right)\right)} \\
&\le \sum_{\ell_0=1}^{L} (\ell_0+2) \sum_{\Gamma:\Gamma\ni i,|\Gamma|\ge \ell_0} e^{-\bar b\left(\frac{1}{\ell_0^{1-2\alpha}(4+\log \ell_0)}\left(\sum_{k=1}^{k_{\underline{\Gamma}}} n_k(\underline{\Gamma})(4+\log k)+(4+\log \ell_0)\right)\right)}.
\end{aligned}
\tag{4.42}
$$

Take $D>1$ and $g_2(\bar b)>1$ so that

$$
\frac{\bar b}{L^{1-2\alpha}(4+\log L)} \ge D g_2(\bar b). \tag{4.43}
$$

Applying (7.15), if $Dg_2(\bar b)\ge C_0\vee 3$ we get

$$
I_1(i) \le \sum_{\ell_0=1}^{L} (\ell_0+2) e^{-Dg_2(\bar b)(4+\log \ell_0)} \sum_{\ell_2\ge \ell_0} 2\ell_2 e^{-Dg_2(\bar b)(4+\log \ell_2)} \le 10 e^{-8Dg_2(\bar b)}. \tag{4.44}
$$

It remains to consider the second term in (4.21), the third term being identical. Using (4.38), (4.16), and (7.15) for each $i\in V$, we have

$$
\begin{aligned}
I_2(i) &\equiv \sum_{\ell_0=1}^{L} \sum_{\Delta:\Delta\ni i,\,|\Delta|=\ell_0} \sum_{\ell_1=1}^{\ell_0} \sum_{T_{left}:|T_{left}|=\ell_1} 1\!\!1_{\{T_{left}\,\triangleleft\,\Delta\}} \sum_{\underline{T}\in\mathcal{A}_2(\ell_1)\setminus T_{left}} I\!\!E\left[\mu_\Lambda^+(T_{left}\cup \underline{T})\right] \\
&\le \sum_{\ell_0=1}^{L} \ell_0 \sum_{\ell_1=1}^{\ell_0} (\ell_1+2) e^{-Dg_2(\bar b)(4+\log \ell_1)} \sum_{\Gamma:\Gamma\ni 0;|\Gamma|\ge \ell_1} e^{-Dg_2(\bar b)(4+\log|\Gamma|)} \\
&\le 5e^{-8Dg_2(\bar b)} \sum_{\ell_0=1}^{L} \ell_0 \le 5L^2 e^{-8Dg_2(\bar b)}.
\end{aligned}
\tag{4.45}
$$

Collecting (4.44) and (4.45) one gets

$$
I\!\!E\left[\mu_\Lambda^+(\nu_i(L,-))\right] \le 20L^2 e^{-8Dg_2(\bar b)}. \tag{4.46}
$$

By Markov inequality, on a probability subset $\Omega_4=\Omega_4(L,i)$ with

$$
I\!\!P[\Omega(L,i)] \ge 1-5Le^{-4Dg_2(\bar b)}, \tag{4.47}
$$

one gets

$$
\mu_\Lambda^+(\nu_i(L,-)) \le 5Le^{-4Dg_2(\bar b)}.
$$

Recalling the definition of $\mathcal{V}(V,\ell_0)$ see (4.3), one gets that on a probability subset $\Omega_5=\Omega_5(V)$ with

$$
I\!\!P[\Omega_5] \ge 1-|V|5Le^{-4Dg_2(\bar b)} \tag{4.48}
$$

we have

$$
\mu_\Lambda^+(\mathcal{V}(V,L) \le 5|V|Le^{-4Dg_2(\bar b)}. \tag{4.49}
$$

Choice of the parameters

• $0 < \alpha < \frac{1}{2}$. From (4.43) we take

$$L \equiv L_{\min} = \left(\bar b\right)^{\frac{1}{1-2\alpha}} \left(4 + \log\left(\bar b\right)^{\frac{1}{1-2\alpha}}\right)^{-\frac{1}{1-2\alpha}} \left(D g_2(\bar b)\right)^{-\frac{1}{1-2\alpha}} . \tag{4.50}$$

It is easy to check that there exists a $\theta_0 = \theta_0(\alpha)$ and β_0 that depend on α but not on $D > 1$ nor on $g_2(\bar b) \geq 1$ such that (4.43) is satisfied for all $0 < \theta \leq \theta_0$ and all $\beta \geq \beta_0$.

Then one can take the volume V with a diameter similar to (3.6), namely

$$V_{min}(\alpha) = e^{g_2(\bar b)} \left(\bar b\right)^{\frac{1}{1-2\alpha}} . \tag{4.51}$$

An easy computation gives (4.5) and (4.8).

• $\alpha = 0$. Going back to (4.22), the modifications are the following : each time a k^α, respectively an ℓ^α, appears replace it by $(4 + \log k)$, respectively by $(4 + \log \ell)$. The event in the step III, are modified in the same way. The only difference comes with (4.33) replaced by

$$I\!\!E\left[1\!\!1_{B_j}\right] \leq e^{-\bar b\left(\sum_{k=j+1}^{k_{\underline{S}}} n_k(\underline{S}) \frac{(4+\log k)^2}{k} + \frac{(4+\log \ell_0)^2}{\ell_0}\right)} . \tag{4.52}$$

Then (4.41) is modified using

$$\frac{(4 + \log k)^2}{k} \geq \frac{(4 + \log \ell_0)}{\ell_0}(4 + \log k). \tag{4.53}$$

The assumption (4.43) becomes

$$\bar b\ \frac{4 + \log L}{L} \geq D g_2(\bar b). \tag{4.54}$$

Then everything but the choice of L goes as before. Here we choose

$$L \equiv L_{\min} = \frac{\bar b}{D g_2(\bar b)} \left(4 + \log\left[\frac{\bar b}{D g_2(\bar b)}\right]\right) \tag{4.55}$$

and it is easy to see that if $\bar b \geq D g_2(\bar b)$ then (4.54) is satisfied. Then as before taking

$$V_{min}(0) = e^{g_2(\bar b)} \frac{\bar b}{D g_2(\bar b)} \left(4 + \log\left[\frac{\bar b}{D g_2(\bar b)}\right]\right) \tag{4.56}$$

one gets (4.5) and (4.8) after easy estimates.

• $\alpha = 1/2$. (4.33) holds in the following form

$$I\!\!E\left[1\!\!1_{B_j}\right] \leq e^{-\bar b\left(1+\sum_{k=j+1}^{k_{\underline{S}}} n_\ell(\underline{S})\right)} . \tag{4.57}$$

Since $1 + \sum_{k=1}^{\ell} n_k(\underline{S}) \geq 1$ the inequality (4.38) becomes

$$I\!\!E[\mu_\Lambda^+(T \cup \underline{S})] \leq (k_{\underline{S}} + 2) e^{-\frac{\bar b}{2}} e^{-\frac{\bar b}{2}\left(1+\sum_{k=1}^{k_{\underline{S}}} n_k(\underline{S})\right)} . \tag{4.58}$$

The condition (4.43) becomes

$$\frac{\bar b}{2(4 + \log L)} \geq D \geq C_0 \tag{4.59}$$

where C_0 is defined in 7.4. Taking

$$L \equiv L_{min} = e^{\frac{\bar{b}}{2D}-4} \tag{4.60}$$

one has

$$I\!E[\mu_\Lambda^+(\nu_i(L,-))] \leq 20e^{+\frac{\bar{b}}{2D}-8}e^{-\frac{\bar{b}}{2}} \leq 20e^{-\frac{\bar{b}}{2}(1-\frac{1}{D})}. \tag{4.61}$$

Therefore if one takes

$$V_{\min}(1/2) = \frac{1}{20}e^{\frac{\bar{b}}{2}(1-\frac{1}{D})}e^{-2g_2(\bar{b})} \tag{4.62}$$

one gets

$$\mu_\Lambda^+(\mathcal{V}(V_{\min}, L_{\min})) \leq e^{-g_2(\bar{b})} \tag{4.63}$$

with a $I\!P$–probability larger than $1 - e^{-g_2(\bar{b})}$. ■

7 Appendix: Geometrical description of the spin configurations

We will follow the geometrical description of the spin configuration presented in [5] and use the same notations. We will consider homogeneous boundary conditions, i.e the spins in the boundary conditions are either all $+1$ or all -1. Actually we will restrict ourself to $+$ boundary conditions and consider spin configurations $\sigma = \{\sigma_i, i \in Z\!\!\!Z\} \in \mathcal{X}_+$ so that $\sigma_i = +1$ for all $|i|$ large enough.

In one dimension an interface at $(x, x+1)$ means $\sigma_x \sigma_{x+1} = -1$. Due to the above choice of the boundary conditions, any $\sigma \in \mathcal{X}_+$ has a finite, even number of interfaces. The precise location of the interface is immaterial and this fact has been used to choose the interface points as follows: For all $x \in Z\!\!\!Z$ so that $(x, x+1)$ is an interface take the location of the interface to be a point inside the interval $[x+\frac{1}{2}-\frac{1}{100}, x+\frac{1}{2}+\frac{1}{100}]$, with the property that for any four distinct points r_i, $i = 1, \ldots, 4$ $|r_1 - r_2| \neq |r_3 - r_4|$. This choice is done once for all so that the interface between x and $x+1$ is uniquely fixed. Draw from each one of these interfaces points two lines forming respectively an angle of $\frac{\pi}{4}$ and of $\frac{3}{4}\pi$ with the $Z\!\!\!Z$ line. We have thus a bunch of growing $\vee-$ lines each one emanating from an interface point. Once two $\vee-$ lines meet, they are frozen and stop their growth. The other two lines emanating from the the same interface points are erased. The $\vee-$ lines emanating from others points keep growing. The collision of the two lines is represented graphically by a triangle whose basis is the line joining the two interfaces points and whose sides are the two segment of the $\vee-$ lines which meet. The choice done of the location of the interface points ensure that collisions occur one at a time so that the above definition is unambiguous. In general there might be triangles inside triangles. The endpoints of the triangles are suitable coupled pairs of interfaces points. The graphical representation just described maps each spin configuration in $\mathcal{X}_+$ to a set of triangles.

Notation *Triangles will be usually denoted by T, the collection of triangles constructed as above by $\mathcal{T}$ and we will write*

$$|T| = \textit{cardinality of} \cap Z\!\!\!Z = \textit{mass of} \quad T, \tag{7.1}$$

and by supp$(T) \subset I\!R$ the basis of the triangle.

We have thus represented a configuration $\sigma \in \mathcal{X}_+$ as a collection of $\underline{T} = (T_1, \ldots, T_n)$. The above construction defines a one to one map from $\mathcal{X}_+$ onto $\mathcal{T}$. It is easy to see that a triangle configuration $\underline{T}$ belongs to $\mathcal{T}$ iff for any pair T and T' in $\underline{T}$

$$\mathrm{dist}(T, T') \geq \min\{|T|, |T'|\}. \tag{7.2}$$

We say that two collections of triangles $\underline{S}'$ and $\underline{S}$ are compatible and we denote it by $\underline{S}' \sim \underline{S}$ iff $\underline{S}' \cup \underline{S} \in \mathcal{T}$ (*i.e.* there exists a configuration in $\mathcal{X}_+$ such that its corresponding collection of triangles is the collection made of all triangles that are obtained by concatenating $\underline{S}'$ and $\underline{S}$.) By an abuse of notation, we write

$$H_0^+(\underline{T}) = H_0^+(\sigma), \quad G(\sigma(\underline{T}))[\omega] = G(\sigma)[\omega], \quad \sigma \in \mathcal{X}_+ \iff \underline{T} \in \mathcal{T}$$

.

Definition 7.1 The energy difference *Given two compatible collections of triangles $\underline{S} \sim \underline{T}$, we denote*

$$H^+(\underline{S}|\underline{T}) := H^+(\underline{S} \cup \underline{T}) - H^+(\underline{T}). \tag{7.3}$$

Let $\underline{T} = (T_1, \dots, T_n)$ with $|T_i| \leq |T_{i+1}|$ then using (7.3) one has

$$H^+(\underline{T}) = H^+(T_1|\underline{T} \setminus T_1) + H^+(\underline{T} \setminus T_1). \tag{7.4}$$

The following Lemma proved in [5], see Lemma 2.1 there, gives a lower bound on the cost to "erase" triangles sequentially starting from the smallest ones.

Lemma 7.2 [5] *For $\alpha \in [0, \frac{\ln 3}{\ln 2} - 1)$ and*

$$\zeta = \zeta(\alpha) = 1 - 2(2^\alpha - 1) \tag{7.5}$$

one has

$$H_0^+(T_1|\underline{T} \setminus T_1) \geq \zeta |T_1|^\alpha. \tag{7.6}$$

By iteration, for any $1 \leq i \leq n$

$$H_0^+(\cup_{\ell=1}^i T_\ell|\underline{T} \setminus [\cup_{\ell=1}^i T_\ell]) \geq \zeta \sum_{\ell=1}^i |T_\ell|^\alpha. \tag{7.7}$$

For $\alpha = 0$, (7.6) and (7.7) hold with $|T_\ell|^\alpha$ replaced by $\log |T_\ell| + 4$.

The estimate (7.7) involves contributions coming from the full set of triangles associated to a given spin configuration, starting from the triangle having the smallest mass. To implement a Peierls bound in our set up we need to "localize" the estimates to compute the weight of a triangle or of a finite set of triangles in a generic configuration. In order to do this [5] introduced the notion of contours as clusters of nearby triangles sufficiently far away from all other triangles.

Contours A contour Γ is a collection $\underline{T}$ of triangles related by a hierarchical network of connections controlled by a positive number C, see (7.8), under which all the triangles of a contour become mutually connected. We denote by $T(\Gamma)$ the triangle whose basis is the smallest interval which contains all the triangles of the contour. The right and left endpoints of $T(\Gamma) \cap \mathbb{Z}$ are denoted by $x_\pm(\Gamma)$. We denote $|\Gamma|$ the mass of the contour Γ

$$|\Gamma| = \sum_{T \in \Gamma} |T|$$

i.e. $|\Gamma|$ is the sum of the masses of all the triangles belonging to Γ. We denote by $\mathcal{R}(\cdot)$ the algorithm which associates to any configuration $\underline{T}$ a configuration $\{\Gamma_j\}$ of contours with the following properties.

P.0 *Let* $\mathcal{R}(\underline{T}) = (\Gamma_1, \dots, \Gamma_n)$, $\Gamma_i = \{T_{j,i}, 1 \le j \le k_i\}$, *then* $\underline{T} = \{T_{j,i}, 1 \le i \le n, 1 \le j \le k_i\}$

P.1 *Contours are well separated from each other.* Any pair $\Gamma \neq \Gamma'$ verifies one of the following alternatives.

$$T(\Gamma) \cap T(\Gamma') = \emptyset$$

i.e. $[x_-(\Gamma), x_+(\Gamma)] \cap [x_-(\Gamma'), x_+(\Gamma')] = \emptyset$, in which case

$$dist(\Gamma, \Gamma') := \min_{T \in \Gamma, T' \in \Gamma'} dist(T, T') > C \min\left\{|\Gamma|^3, |\Gamma'|^3\right\} \tag{7.8}$$

where C is a positive number. If

$$T(\Gamma) \cap T(\Gamma') \neq \emptyset,$$

then either $T(\Gamma) \subset T(\Gamma')$ or $T(\Gamma') \subset T(\Gamma)$; moreover, supposing for instance that the former case is verified, (in which case we call Γ an inner contour) then for any triangle $T'_i \in \Gamma'$, either $T(\Gamma) \subset T'_i$ or $T(\Gamma) \cap T'_i = \emptyset$ and

$$dist(\Gamma, \Gamma') > C|\Gamma|^3, \quad \text{if} \quad T(\Gamma) \subset T(\Gamma'). \tag{7.9}$$

P.2 *Independence.* Let $\{\underline{T}^{(1)}, \dots, \underline{T}^{(k)}\}$, be $k > 1$ configurations of triangles; $\mathcal{R}(\underline{T}^{(i)}) = \{\Gamma_j^{(i)}, j = 1, \dots, n_i\}$ the contours of the configurations $\underline{T}^{(i)}$. Then if any distinct $\Gamma_j^{(i)}$ and $\Gamma_{j'}^{(i')}$ satisfies **P.1**,

$$\mathcal{R}(\underline{T}^{(1)}, \dots, \underline{T}^{(k)}) = \{\Gamma_j^{(i)}, j = 1, \dots, n_i; i = 1, \dots, k\}.$$

As proven in [5], the algorithm $\mathcal{R}(\cdot)$ having properties **P.0**, **P.1** and **P.2** is unique and therefore there is a bijection between families of triangles and contours. Next we report the estimates proven in Theorem 3.2 of [5] which are essential for this paper.

Theorem 7.3 [5] *Let* $\alpha \in [0, \frac{\ln 3}{\ln 2} - 1)$ *and the constant* C *given in (7.8), be so large that*

$$\sum_{m \ge 1} \frac{4m}{[Cm]^3} \le \frac{1}{2}, \tag{7.10}$$

where $[x]$ *denotes the integer part of* x*. For any* $\underline{T} \in \{\underline{T}\}$*, let* $\Gamma_0 \in \mathcal{R}(\underline{T})$ *be a contour,* $\underline{S}^{(0)}$ *the triangles in* Γ_0 *and* $\zeta = \zeta(\alpha) = 1 - 2(2^\alpha - 1)$*. Then*

$$H_0^+(\underline{S}^{(0)} | \underline{T} \setminus \underline{S}^{(0)}) \ge \frac{\zeta}{2} \sum_{T \in \underline{S}^{(0)}} |T|^\alpha. \tag{7.11}$$

For $\alpha = 0$*, (7.11) holds with* $|T|^\alpha$ *replaced by* $\log|T| + 4$ *.*

Next we summarize the results of Theorem 4.1 of [5] stated for $\alpha > 0$ and the corresponding estimate for $\alpha = 0$ given in Appendix F of [5].

Theorem 7.4 [5] *For any* $\alpha > 0$ *there exists* $C_0(\alpha)$ *so that for* $b \ge C_0(\alpha)$ *and for all* $m > 0$

$$\sum_{\{0 \in \Gamma, |\Gamma| = m\}} w_b^\alpha(\Gamma) \le 2m e^{-bm^\alpha}, \tag{7.12}$$

where

$$w_b^\alpha(\Gamma) := \prod_{T\in\Gamma} e^{-b|T|^\alpha}. \tag{7.13}$$

When $\alpha = 0$

$$w_b^0(\Gamma) := \prod_{T\in\Gamma} e^{-b(\log|T|+4)} = \prod_{T\in\Gamma} \left(|T|^{-b} e^{-4b}\right) \tag{7.14}$$

and there exists C_0 *so that for* $b \geq C_0$

$$\sum_{\{0\in\Gamma, |\Gamma|=m\}} w_b^0(\Gamma) \leq 2m e^{-b(\log m+4)}. \tag{7.15}$$

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